

INTERSECTION NUMBERS ON DELIGNE-MUMFORD MODULI SPACES AND QUANTUM AIRY CURVE

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ABSTRACT. We establish the Airy curve case of a conjecture of Gukov and Sulkowski by reducing to Dijkgraaf-Verlinde-Verlinde Virasoro constraints satisfied by the intersection numbers on moduli spaces of algebraic curves.

Key words. Intersection numbers, moduli spaces of curves, Eynard-Orantin topological recursion.

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1. INTRODUCTION

In this paper we will establish the Airy curve case of a conjecture of Gukov and Sulkowski [5]. By the Airy curve we mean the plane algebraic curve defined by the following equation:

$$(1) \quad A(u, v) = \frac{1}{2}v^2 - u = 0.$$

(This differs from the form used in [5] by a factor of 2.) This curve has the following parametrization:

$$(2) \quad u(p) = \frac{1}{2}p^2,$$

$$(3) \quad v(p) = p.$$

By the Eynard-Orantin recursion [3], one can define from this curve a family of differentials:

$$(4) \quad W_{g,n}(p_1, \dots, p_n) = \mathcal{W}_{g,n}(p_1, \dots, p_n) dp_1 \cdots dp_n.$$

Motivated by the matrix model origin of this construction, or a definition of the Baker-Akhiezer function in [3], Gukov and Sulkowski [5] define

$$(5) \quad Z = \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n,$$

where S_n are defined by:

$$(6) \quad S_0 = \int^p v(p) du(p),$$

$$(7) \quad S_1(p) = -\frac{1}{2} \log \frac{du}{dp},$$

$$(8) \quad S_n(p) = \sum_{2g-1+k} \frac{(-1)^k}{k!} \int^p \cdots \int^p W_{g,k}(p'_1, \dots, p'_k) dp'_1 \cdots dp'_k, \quad n \geq 2.$$

We use a different sign convention from that in [5]. We will prove the following:

Theorem 1.1. *The function Z satisfies the following differential equation:*

$$(9) \quad \hat{A}(u, v)Z = 0,$$

where

$$(10) \quad \hat{A} = \frac{1}{2} \hat{v}^2 - \hat{u}$$

is the quantization of the polynomial $A(u, v)$, where $\hat{u} = u \cdot$, $\hat{v} = \hbar \partial_u$.

This is a special case of the general conjecture made by Gukov-Sulkowski [5] on quantizable algebraic curves. We prove this result by reducing to the Dijkgraaf-Verlinde-Verlinde recursion relation [2]. In a subsequent work [9], we will treat the case of the local mirror curve for \mathbb{C}^3 and the resolved conifold.

In [4, §10], Eynard and Orantin claimed that for the curve

$$(11) \quad x(z) = z^2,$$

$$(12) \quad y(z) = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{k+2} z^k,$$

the differentials $W_{g,n}(z_1, \dots, z_n)$ encodes higher Weil-Petersson volumes, and one can also construct F_g that encodes intersection numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g$. See also [3, §10.4.1]. In [3, §10], the Eynard-Orantin recursion for the Airy curve was discussed and the relationship between $W_{g,n}$ the Tracy-Widom kernel was recalled. It was also stated that “The fact that the Baker-Akhiezer function is $Ai(x)$ and satisfies the differential equation $Ai'' = xAi$ can be seen as a consequence of the Hirota equation theorem 9.2.” The authors of [1] established the equivalence of the Eynard-Orantin recursion of the Airy curve to the DVV recursion relations via the Laplace transform of a recursion relation for the symplectic volumes of the moduli spaces. In [8], the author established the equivalence of the DVV recursion relations to a

Eynard-Orantin type recursions, but unfortunately, the starting point of that paper was to rewrite the DVV recursion relation using residues and a kernel function, not necessarily on an algebraic curve, so the relationship with the Airy curve was missed. After the author came across [1] in June 2012 on the internet, it became clear that it is possible to combine the ideas in [1] and [8] to directly establish the equivalence between the DVV relations and the Eynard-Orantin recursion for the Airy curve. A consequence of this result and Theorem 1.1 is that one can then relate the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ to the Airy functions $Ai(x)$ and $Bi(x)$.

The rest of the paper is arranged as follows. In §2 we recall the Eynard-Orantin recursion for the Airy curve and present a direct proof that it is equivalent to the DVV recursion. We then change coordinate in §3 and combine an observation in an earlier work [8] to derive a simple recursion relation for a suitably defined n -point polynomial functions

$$\omega_{g,n} = \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^n (2a_i + 1)!! w_i^{a_i+1}$$

of intersection numbers on the Deligne-Mumford moduli spaces. It has a very simple form:

$$\begin{aligned} & \omega_{g,n+1}(w_0, w_1, \dots, w_n) \\ &= \frac{1}{2} w_0 \omega_{g-1,n+2}(w_0, w_0, w_{[n]}) \\ &+ \frac{1}{2} w_0 \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2=[n]}}^s \omega_{g_1,|A_1|+1}(w_0, w_{A_1}) \cdot \omega_{g_2,|A_2|+1}(w_0, w_{A_2}) \\ &+ \sum_{i=1}^n D_{w_0, w_i} \omega_{g,n}(x, w_{[n]_i}), \end{aligned}$$

where

$$D_{u,v} x^m = uv(u^m + 3u^{m-1}v + 5u^{m-2}v^2 + \cdots + (2m+1)v^m).$$

In §4 we consider the antiderivatives $\Omega_{g,n}$ of $\omega_{g,n}$:

$$\Omega_{g,n}(w_1, \dots, w_n) = \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^n (2a_i - 1)!! w_i^{a_i+1/2}$$

and integrate the above recursion relation to get a recursion relation

$$\begin{aligned}
& \partial_{w_0} \Omega_{g,n+1}(w_0, w_1, \dots, w_n) \\
&= w_0^{5/2} \partial_x \partial_y \Omega_{g-1,n+2}(x, y, w_{[n]})|_{x=y=w_0} \\
&+ w_0^{5/2} \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2=[n]}}^s \partial_{w_0} \Omega_{g_1,|A_1|+1}(w_0, w_{A_1}) \cdot \partial_{w_0} \Omega_{g_2,|A_2|+1}(w_0, w_{A_2}) \\
&+ w_0^{-3/2} \sum_{i=1}^n \mathcal{D}_{w_0, w_i} \partial_x \Omega_{g,n}(x, w_{[n]_i}),
\end{aligned}$$

where $\mathcal{D}_{u,v} : \mathbb{C}[x]x^{-1/2} \rightarrow \mathbb{C}[u,v]uv^{1/2}$ is a linear operator defined by:

$$\mathcal{D}_{u,v} x^{a-1/2} = uv^{1/2}(u^{a+1} + u^a v + \dots + v^{a+1}).$$

In §5 we use this result in §4 to present a proof of Theorem 1.1.

2. EYNARD-ORANTIN TOPOLOGICAL RECURSION ON AIRY CURVE AND MODULI SPACES OF CURVES

2.1. Eynard-Orantin recursion for the Airy curve. In this section, we recall the construction of Eynard-Orantin [3, §4] for the case of the Airy curve. Near the branch point $(u, v) = 0$, the conjugate point of $(u(p), v(p))$ is $(u(p), -v(p))$, i.e. $\bar{p} = -p$. Hence the vertex is given by:

$$(13) \quad \omega = (v(\bar{p}) - v(p))du(p) = ((-p) - p)d\frac{p^2}{2} = -2p^2 dp.$$

Since the Airy curve has genus 0, under the parametrization (2) and (3), the line-propagator (Bergmann kernel) on the Airy curve is given by [3, §3.2]:

$$(14) \quad B(p_1, p_2) = \frac{dp_1 dp_2}{(p_1 - p_2)^2}.$$

The arrow-propagator is

$$(15) \quad dE_q(p) = \frac{1}{2} \int_q^{\bar{q}} B(\xi, p) = \frac{1}{2} dp \int_q^{-q} \frac{d\xi}{(p - \xi)^2} = \frac{qdp}{q^2 - p^2}.$$

Hence the recursion kernel is:

$$(16) \quad K(q, p) = \frac{dE_q(p)}{\omega(q)} = \frac{dp}{q(q^2 - p^2)dq}.$$

The Eynard-Orantin recursion has as initial values:

$$(17) \quad W_{0,1}(p) = 0,$$

$$(18) \quad W_{0,2}(p_1, p_2) = B(p_1, p_2) = \frac{dp_1 dp_2}{(p_1 - p_2)^2},$$

and in general:

$$(19) \quad \begin{aligned} & W_{g,n+1}(z_0, z_1, \dots, z_n) \\ &= \frac{1}{2} \operatorname{Res}_{z=0} \left(K(z, z_0) \cdot \left(W_{g-1,n+2}(z, -z, z_{[n]}) \right. \right. \\ & \quad \left. \left. + \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2=[n]}} W_{g_1,|A_1|+1}(z, z_{A_1}) \cdot W_{g_2,|A_2|+1}(-z, z_{A_2}) \right) \right). \end{aligned}$$

Here we have used the following notations: For $A \subset [n]$, when $A = \emptyset$, z_A is empty; otherwise, if $A = \{i_1, \dots, i_k\}$, then $z_A = z_{i_1}, \dots, z_{i_k}$. In terms of $\mathcal{W}_{g,n}(z_1, \dots, z_n)$, one has

$$(20) \quad \begin{aligned} & \mathcal{W}_{g,n+1}(z_0, z_1, \dots, z_n) \\ &= \frac{1}{2} \operatorname{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot \left(\mathcal{W}_{g-1,n+2}(z, -z, z_{[n]}) \right. \right. \\ & \quad \left. \left. + \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2=[n]}} \mathcal{W}_{g_1,|A_1|+1}(z, z_{A_1}) \cdot \mathcal{W}_{g_2,|A_2|+1}(-z, z_{A_2}) \right) \right). \end{aligned}$$

Here are some examples obtained by applying (20).

$$\begin{aligned} & \mathcal{W}_{0,3}(z_0, z_1, z_2) \\ &= \operatorname{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \mathcal{W}_{0,2}(z, z_1) \mathcal{W}_{0,2}(z, z_2) \right) \\ &= \operatorname{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot \frac{1}{(z - z_1)^2} \cdot \frac{1}{(z - z_2)^2} \right) \\ &= \frac{1}{z_0^2 z_1^2 z_2^2}. \end{aligned}$$

$$\begin{aligned}
& \mathcal{W}_{0,4}(z_0, z_1, z_2, z_3) \\
&= \text{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot (\mathcal{W}_{0,2}(z, z_1)\mathcal{W}_{0,3}(z, z_2, z_3) \right. \\
&+ \mathcal{W}_{0,2}(z, z_2)\mathcal{W}_{0,3}(z, z_1, z_3) + \mathcal{W}_{0,2}(z, z_3)\mathcal{W}_{0,3}(z, z_1, z_2)) \\
&= \text{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot \left(\frac{1}{(z - z_1)^2} \cdot \frac{1}{z^2 z_2^2 z_3^2} \right. \right. \\
&+ \left. \left. \frac{1}{(z - z_2)^2} \cdot \frac{1}{z^2 z_1^2 z_3^2} + \frac{1}{(z - z_3)^2} \cdot \frac{1}{z^2 z_1^2 z_2^2} \right) \right).
\end{aligned}$$

Because

$$\text{Res}_{z=0} \frac{1}{z^3(u^2 - z^2)(z - v)^2} = \frac{3}{u^2 v^4} + \frac{1}{u^4 v^2},$$

we have

$$\mathcal{W}_{0,4}(z_0, z_1, z_2, z_3) = \frac{1}{z_0^2 z_1^2 z_2^2 z_3^2} \sum_{i=0}^3 \frac{3}{z_i^2}.$$

When $(g, n) = (1, 2)$ we have

$$\begin{aligned}
& \mathcal{W}_{1,2}(z_0, z_1) \\
&= \text{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \left(\frac{1}{2} \mathcal{W}_{0,3}(z, z, z_1) + \mathcal{W}_{0,2}(z, z_1) \mathcal{W}_{1,1}(z) \right) \right) \\
&= \text{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot \left(\frac{1}{2} \cdot \frac{1}{z^4 z_1^2} + \frac{1}{(z - z_1)^2} \cdot \frac{1}{8z^4} \right) \right) \\
&= \frac{1}{2} \cdot \frac{1}{z_0^6 z_1^2} + \frac{1}{8} \left(\frac{5}{z_0^2 z_1^6} + \frac{3}{z_0^4 z_1^4} + \frac{1}{z_0^6 z_1^2} \right) \\
&= \frac{5}{8} \cdot \frac{1}{z_0^2 z_1^6} + \frac{3}{8} \cdot \frac{1}{z_0^4 z_1^4} + \frac{5}{8} \cdot \frac{1}{z_0^6 z_1^2}.
\end{aligned}$$

2.2. Relationship with intersection numbers on moduli spaces of curves. Consider the intersection numbers on Deligne-Mumford moduli spaces:

$$(21) \quad \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g := \int_{\mathcal{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}$$

and the following generating function

$$(22) \quad \tilde{\mathcal{W}}_{g,n}(z_1, \dots, z_n) = \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^n \frac{(2a_i + 1)!!}{z_i^{2a_i + 2}}.$$

By evaluating the correlators by Witten-Kontsevich Theorem [7, 6], one can find explicit expressions of $\tilde{\mathcal{W}}_{g,n}$ for small g and n . For example,

$$\begin{aligned}\tilde{\mathcal{W}}_{0,3}(z_1, z_2, z_3) &= \frac{1}{z_1^2 z_2^2 z_3^2}, \\ \tilde{\mathcal{W}}_{0,4}(z_1, z_2, z_3, z_4) &= \frac{1}{z_1^2 z_2^2 z_3^2 z_4^2} \sum_{i=1}^4 \frac{3}{z_i^2}, \\ \tilde{\mathcal{W}}_{0,5}(z_1, \dots, z_5) &= \frac{1}{z_1^2 \dots z_5^2} \left(\sum_{i=1}^5 \frac{15}{z_i^4} + \sum_{1 \leq i < j \leq 5} \frac{18}{z_i^2 z_j^2} \right), \\ \tilde{\mathcal{W}}_{0,6}(z_1, \dots, z_6) &= \frac{1}{\prod_{i=1}^6 z_i^2} \left(\sum_{i=1}^6 \frac{105}{z_i^6} + \sum_{1 \leq i \neq j \leq 6} \frac{135}{z_i^4 z_j^2} + \sum_{1 \leq i < j < k \leq 6} \frac{162}{z_i^2 z_j^2 z_k^2} \right),\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{W}}_{1,1}(z_1) &= \frac{1}{8z_1^4}, \\ \tilde{\mathcal{W}}_{1,2}(z_1, z_2) &= \frac{1}{8z_1^2 z_2^2} \left(\frac{5}{z_1^4} + \frac{5}{z_2^4} + \frac{3}{z_1^2 z_2^2} \right), \\ \tilde{\mathcal{W}}_{1,3}(z_1, z_2, z_3) &= \frac{1}{8z_1^2 z_2^2 z_3^2} \cdot \left(\sum_{i=1}^3 \frac{35}{z_i^6} + \sum_{1 \leq i \neq j \leq 3} \frac{30}{z_i^4 z_j^2} + \frac{18}{z_1^2 z_2^2 z_3^2} \right),\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{W}}_{2,1}(z_1) &= \frac{105}{128} \frac{1}{z_1^{10}}, \\ \tilde{\mathcal{W}}_{2,2}(z_1, z_2) &= \frac{1155}{128} \frac{1}{z_1^{12} z_2^2} + \frac{3465}{128} \frac{1}{z_1^{10} z_2^4} + \frac{6699}{128} \frac{1}{z_1^8 z_2^6} + \frac{6699}{128} \frac{1}{z_1^6 z_2^8} \\ &\quad + \frac{3465}{128} \frac{1}{z_1^4 z_2^{10}} + \frac{1155}{128} \frac{1}{z_1^2 z_2^{12}}\end{aligned}$$

etc.

Theorem 2.1. *When $2g - 2 + n > 0$, one has*

$$(23) \quad \mathcal{W}_{g,n}(z_1, \dots, z_n) = \tilde{\mathcal{W}}_{g,n}(z_1, \dots, z_n).$$

Indeed, $\tilde{\mathcal{W}}_{g,n}$'s satisfy the initial values and Eynard-Orantin recursion relations (17)-(19).

As mentioned in the Introduction, this result was due to [1]. In that work, the result was established via an equivalent recursion relations for the symplectic volumes of the moduli spaces [1, Theorem 1.1]. Here we present a direct proof.

Proof. Recall the DVV recursion relations [2] are:

$$\begin{aligned}
 \langle \tilde{\tau}_{a_0} \prod_{i=1}^n \tilde{\tau}_{a_i} \rangle_g &= \sum_{i=1}^n (2a_i + 1) \langle \tilde{\tau}_{a_0+a_i-1} \prod_{j \in [n]_i} \tilde{\tau}_{a_j} \rangle_g \\
 (24) \quad &+ \frac{1}{2} \sum_{b_1+b_2=a_0-2} \left(\langle \tilde{\tau}_{b_1} \tilde{\tau}_{b_2} \prod_{i=1}^n \tilde{\tau}_{a_i} \rangle_{g-1} \right. \\
 &\left. + \sum_{\substack{A_1 \amalg A_2=[n] \\ g_1+g_2=g}}^s \langle \tilde{\tau}_{b_1} \prod_{i \in A_1} \tilde{\tau}_{a_i} \rangle_{g_1} \cdot \langle \tilde{\tau}_{b_2} \prod_{i \in A_2} \tilde{\tau}_{a_i} \rangle_{g_2} \right),
 \end{aligned}$$

where $\tilde{\tau}_a = (2a+1)!! \cdot \tau_a$ and $[n] = \{1, \dots, n\}$, $[n]_i = [n] - \{i\}$. In the above formula,

$$\sum_{\substack{A_1 \amalg A_2=[n] \\ g_1+g_2=g}}^s$$

means the summation is taken over the “stable cases”, i.e.:

$$2g_1 - 1 + |A_1| > 0, \quad 2g_2 - 1 + |A_2| > 0.$$

Multiply both sides of (24) by $\frac{1}{z_0^{2a_0+2}} \cdot \prod_{i=1}^n \frac{1}{z_i^{2a_i+2}}$ and take summations over a_0, a_1, \dots, a_n :

$$\begin{aligned}
 &\tilde{\mathcal{W}}_{g,n+1}(z_0, z_1, \dots, z_n) \\
 &= \sum_{a_0, a_1, \dots, a_n \geq 0} \langle \tau_{a_0} \tau_{a_1} \cdots \tau_{a_n} \rangle_g \frac{(2a_0+1)!!}{z_0^{2a_0+2}} \cdot \prod_{i=1}^n \frac{(2a_i+1)!!}{z_i^{2a_i+2}} \\
 &= \sum_{a_0, a_1, \dots, a_n \geq 0} \sum_{i=1}^n (2a_i+1) \cdot (2a_0+2a_i-1)!! \\
 &\quad \cdot \langle \tau_{a_0+a_i-1} \prod_{j \in [n]_i} \tau_{a_j} \rangle_g \cdot \frac{1}{z_0^{2a_0+2}} \cdot \frac{1}{z_i^{2a_i+2}} \cdot \prod_{j \in [n]_i} \frac{(2a_j+1)!!}{z_j^{2a_j+2}} \\
 &+ \frac{1}{2} \sum_{b_1+b_2=a_0-2} (2b_1+1)!! \cdot (2b_2+1)!! \left(\langle \tau_{b_1} \tau_{b_2} \prod_{i=1}^n \tau_{a_i} \rangle_{g-1} \right. \\
 &+ \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2}}^s \langle \tau_{b_1} \prod_{i \in A_1} \tau_{a_i} \rangle_{g_1} \cdot \langle \tau_{b_2} \prod_{i \in A_2} \tau_{a_i} \rangle_{g_2} \Big) \\
 &\quad \cdot \frac{1}{z_0^{2b_1+2b_2+6}} \cdot \prod_{i=1}^n \frac{(2a_i+1)!!}{z_i^{2a_i+2}}.
 \end{aligned}$$

Notice that

$$(25) \quad \text{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot \frac{1}{z^{2b_1+2}} \cdot \frac{1}{z^{2b_2+2}} \right) = \frac{1}{z_0^{2b_1+2b_2+6}},$$

and

$$(26) \quad \text{Res}_{z=0} \left(\frac{1}{z(z_0^2 - z^2)} \cdot \frac{1}{(z - z_i)^2} \cdot \frac{1}{z^{2m+2}} \right) = \sum_{a_0=0}^{m+1-a_0} \frac{(2a_i + 1)}{z_0^{2a_0+2} z_i^{2a_i+2}},$$

so the above equality can be rewritten as follows:

$$\begin{aligned} & \tilde{\mathcal{W}}_{g,n+1}(z_0, z_1, \dots, z_n) \\ &= \sum_{i=1}^n \text{Res}_{z=0} \left(\frac{1}{z(z_0 - z^2)} \cdot \left(\frac{1}{(z - z_i)^2} \cdot \tilde{\mathcal{W}}_{g,n}(-z, [n]_i) \right. \right. \\ &+ \frac{1}{2} \tilde{\mathcal{W}}_{g-1,n+2}(z, -z, [n]) \\ &+ \left. \left. \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2}}^s \tilde{\mathcal{W}}_{g_1,|A_1|+1}(z, z_{A_1}) \cdot \tilde{\mathcal{W}}_{g_2,|A_2|+1}(-z, z_{A_2}) \right) \right). \end{aligned}$$

The proof is completed by setting:

$$\tilde{\mathcal{W}}_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}.$$

□

3. POLYNOMIAL REFORMULATIONS

In this section we will show that in different coordinates the recursion relations in the preceding section can be reformulated as operations on polynomials.

3.1. Change of variable. In §2.1 and 2.2 we have presented some examples of $\mathcal{W}_{g,n}(z_1, \dots, z_n)$. From these examples, it is clear that after the following change of variables

$$w_i = \frac{1}{z_i^2},$$

One gets polynomial expressions:

$$(27) \quad \omega_{g,n}(w_1, \dots, w_n) = \mathcal{W}_{g,n}(z_1, \dots, z_n).$$

By (22),

$$(28) \quad \omega_{g,n}(w_1, \dots, w_n) = \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^n (2a_i + 1)!! w_i^{a_i+1}.$$

The following are some examples.

$$\omega_{0,3}(w_1, w_2, w_3) = w_1 w_2 w_3,$$

$$\omega_{0,4}(w_1, \dots, w_4) = w_1 \cdots w_4 \sum_{i=1}^4 w_i,$$

$$\omega_{0,5}(w_1, \dots, w_5) = w_1 \cdots w_5 (15 \sum_{i=1}^5 w_i^2 + 18 \sum_{1 \leq i < j \leq 5} w_i w_j),$$

$$\begin{aligned} \omega_{0,6}(w_1, \dots, w_6) = & \prod_{i=1}^6 w_i^2 \cdot (105 \sum_{i=1}^6 w_i^3 + 135 \sum_{1 \leq i \neq j \leq 6} w_i^2 w_j \\ & + 162 \sum_{1 \leq i < j < k \leq 6} w_i w_j w_k), \end{aligned}$$

$$\omega_{1,1}(w_1) = \frac{1}{8} w_1^2,$$

$$\omega_{1,2}(w_1, w_2) = \frac{w_1 w_2}{8} (5w_1^2 + 5w_2^2 + 3w_1 w_2),$$

$$\omega_{1,3}(w_1, w_2, w_3) = \frac{w_1 w_2 w_3}{8} \cdot \left(\sum_{i=1}^3 35w_i^3 + \sum_{1 \leq i \neq j \leq 3} 30w_i^2 w_j + 18w_1 w_2 w_3 \right),$$

$$\omega_{2,1}(w_1) = \frac{105}{128} w_1^5,$$

$$\begin{aligned} \omega_{2,2}(w_1, w_2) = & \frac{w_1 w_2}{128} (1155(w_1^5 + w_2^5) + 3465(w_1^4 w_2 + w_1 w_2^4) \\ & + 6699(w_1^3 w_2^2 + w_1^2 w_2^3)). \end{aligned}$$

3.2. Recursion relations for $\omega_{g,n}$.

Theorem 3.1. *Except for the case of $(g, n) = (0, 2)$, the following recursion relations hold:*

$$\begin{aligned} & \omega_{g,n+1}(w_0, w_1, \dots, w_n) \\ &= \frac{1}{2} w_0 \omega_{g-1,n+2}(w_0, w_0, w_{[n]}) \\ (29) \quad & + \frac{1}{2} w_0 \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2=[n]}}^s \omega_{g_1,|A_1|+1}(w_0, w_{A_1}) \cdot \omega_{g_2,|A_2|+1}(w_0, w_{A_2}) \\ & + \sum_{i=1}^n D_{w_0, w_i} \omega_{g,n}(x, w_{[n]_i}), \end{aligned}$$

where for $m \geq 0$,

$$(30) \quad D_{u,v}x^m = uv(u^m + 3u^{m-1}v + 5u^{m-2}v^2 + \cdots + (2m+1)v^m).$$

Proof. By (28), (29) is equivalent to (24). \square

The $(g, n) = (0, 2)$ case is exceptional and can be treated separately as follows.

$$\begin{aligned} & \omega_{0,3}(w_0, w_1, w_2) \\ &= \int_{|u|=1/\epsilon^{1/2}} \frac{w_0}{u-w_0} \cdot \omega_{0,2}(u, w_1) \cdot \omega_{0,2}(u, w_2) du \\ &= \int_{|u|=1/\epsilon^{1/2}} \frac{w_0}{u-w_0} \cdot \frac{1}{(u-w_1)^2} \cdot \frac{1}{(u-w_2)^2} du. \end{aligned}$$

A complicated residue calculation by Maple yields:

$$\omega_{0,3}(w_0, w_1, w_2) = w_0 w_1 w_2.$$

This is a match with (23).

3.3. Examples.

3.3.1. *The $(g, n) = (0, 3)$ case.* This is the first case of (29):

$$\begin{aligned} \omega_{0,4}(w_0, w_1, w_2, w_3) &= \sum_{i=1}^3 D_{w_0, w_i} \omega_{0,3}(x, w_{[3]_i}) \\ &= \sum_{i=1}^3 D_{w_0, w_i} \left(x \cdot \frac{w_1 w_2 w_3}{w_i} \right) = \sum_{i=1}^3 w_0 w_i (w_0 + 3w_i) \cdot \frac{w_1 w_2 w_3}{w_i} \\ &= 3w_0^2 w_1 w_2 w_3 + 3w_0 w_1 w_2 w_3 \sum_{i=1}^3 w_i = 3w_0 w_1 w_2 w_3 \sum_{i=0}^3 w_i. \end{aligned}$$

3.3.2. *The $(g, n) = (1, 1)$ case.*

$$\begin{aligned} \omega_{1,2}(w_0, w_1) &= \frac{1}{2} w_0 \omega_{0,3}(w_0, w_0, w_1) + D_{w_0, w_1} \omega_{1,1}(x) \\ &= \frac{1}{2} w_0 \cdot w_0^2 w_1 + D_{w_0, w_1} \frac{1}{8} x^2 \\ &= \frac{1}{2} w_0^3 w_1 + \frac{1}{8} w_0 w_1 (w_0^2 + 3w_0 w_1 + 5w_1^2) \\ &= \frac{1}{8} w_0 w_1 (5w_0^2 + 5w_1^2 + 3w_0 w_1). \end{aligned}$$

3.4. Derivation of (29). Let us explain how (29) was derived originally. In [8], the author has shown that the DVV recursion relations are equivalent to Eynard-Orantin type recursion with initial value

$$\mathcal{W}_{0,2}(z_1, z_2) = \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2}$$

and kernel function:

$$K(z_1, z_2) = \frac{1}{z_1 z_2 (z_2 - z_1)}.$$

It is easy to check that the same holds for

$$\begin{aligned} \mathcal{W}_2(z_1, z_2) &= \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2}, \\ K(z_1, z_2) &= \frac{1}{z_1(z_2^2 - z_1^2)}. \end{aligned}$$

In the w coordinates, we have

$$\omega_2(w_1, w_2) = \frac{w_1 w_2 (w_1 + w_2)}{(w_1 - w_2)^2} = w_2 + \frac{3w_2^2}{w_1 - w_2} + \frac{2w_2^3}{(w_1 - w_2)^2}.$$

and we have

$$\begin{aligned} & \omega_{g,n+1}(w_0, w_1, \dots, w_n) \\ &= \frac{1}{2} \cdot \frac{1}{2\pi i} \int_{|z|=\epsilon} \left(\frac{1}{z(\frac{1}{w_0} - z^2)} \cdot \left(\omega_{g-1,n+2}\left(\frac{1}{z^2}, \frac{1}{z^2}, w_{[n]}\right) \right. \right. \\ &+ \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}} \omega_{g_1,|A_1|+1}\left(\frac{1}{z^2}, w_{A_1}\right) \cdot \omega_{g_2,|A_2|+1}\left(\frac{1}{z^2}, w_{A_2}\right) \Bigg) dz \\ &= -\frac{1}{2\pi i} \int_{|w|=1/\epsilon^{1/2}} \frac{w^{1/2}}{\frac{1}{w_0} - \frac{1}{w}} \cdot \left(\omega_{g-1,n+2}(w, w, w_{[n]}) \right. \\ &+ \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}} \omega_{g_1,|A_1|+1}(w, w_{A_1}) \cdot \omega_{g_2,|A_2|+1}(w, w_{A_2}) \Bigg) d\frac{1}{w^{1/2}} \\ &= \frac{1}{2} \cdot \frac{1}{2\pi i} \int_{|w|=1/\epsilon^{1/2}} \frac{w_0}{w - w_0} \cdot \left(\omega_{g-1,n+2}(w, w, w_{[n]}) \right. \\ &+ 2 \sum_{i=1}^n \omega_{0,2}(w, w_i) \cdot \omega_{g,n}(w, w_{[n]_i}) \\ &+ \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}}^s \omega_{g_1,|A_1|+1}(w, w_{A_1}) \cdot \omega_{g_2,|A_2|+1}(w, w_{A_2}) \Bigg) dw. \end{aligned}$$

The first line on the right-hand side of the last equality is:

$$(31) \quad I = \frac{1}{2} w_0 \omega_{g-1, n+2}(w_0, w_0, w_{[n]}).$$

The third line on the right-hand side of the last equality is:

$$(32) \quad III = \frac{1}{2} w_0 \sum_{\substack{g_1+g_2=g \\ A_1 \amalg A_2=[n]}}^s \omega_{g_1, |A_1|+1}(w_0, w_{A_1}) \cdot \omega_{g_2, |A_2|+1}(w_0, w_{A_2}).$$

The second line on the right-hand side of the last equality is more complicated. Recall that if $f(z)$ is holomorphic at $z = z_0$, then one has

$$\begin{aligned} \operatorname{Res}_{z=z_0} \frac{f(z)}{z - z_0} &= f(z_0), \\ \operatorname{Res}_{z=z_0} \frac{f(z)}{(z - z_0)^2} &= f'(z_0). \end{aligned}$$

So we have

$$\begin{aligned} II &= \frac{1}{2\pi i} \int_{|w|=\epsilon^{-1/2}} \frac{w_0}{u - w_0} \cdot \sum_{i=1}^n \omega_{0,2}(u, w_i) \cdot \omega_{g,n}(u, w_{[n]_i}) du \\ &= \frac{1}{2\pi i} \int_{|w|=\epsilon^{-1/2}} \frac{w_0}{u - w_0} \cdot \sum_{i=1}^n \left(w_i + \frac{3w_i^2}{u - w_i} + \frac{2w_i^3}{(u - w_i)^2} \right) \cdot \omega_{g,n}(u, w_{[n]_i}) du \\ &= \sum_{i=1}^n w_0 \cdot \frac{w_0 w_i (w_0 + w_i)}{(w_0 - w_i)^2} \cdot \omega_{g,n}(w_0, w_{[n]_i}) + \sum_{i=1}^n \frac{w_0}{w_i - w_0} \cdot 3w_i^2 \cdot \omega_{g,n}(w_{[n]}) \\ &\quad - \sum_{i=1}^n \frac{w_0}{(w_i - w_0)^2} \cdot 2w_i^3 \cdot \omega_{g,n}(w_{[n]}) + \sum_{i=1}^n \frac{w_0}{w_i - w_0} \cdot 2w_i^3 \partial_{w_i} \omega_{g,n}(w_{[n]}). \end{aligned}$$

Now we introduce a linear operator

$$(33) \quad D_{u,v} : \mathbb{C}[x] \rightarrow \mathbb{C}[u, v],$$

defined as follows:

$$D_{u,v} f(x) := uv \cdot \left(\frac{u(u+v)}{(u-v)^2} \cdot f(u) + \frac{3v}{v-u} f(v) - \frac{2v^2}{(v-u)^2} f(v) + \frac{2v^2}{v-u} \cdot f'(v) \right).$$

With this operator, we have:

$$\begin{aligned}
& \omega_{g,n+1}(w_0, w_1, \dots, w_n) \\
&= \frac{1}{2} w_0 \omega_{g-1,n+2}(w_0, w_0, w_{[n]}) \\
&+ \frac{1}{2} w_0 \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}}^s \omega_{g_1,|A_1|+1}(w_0, w_{A_1}) \cdot \omega_{g_2,|A_2|+1}(w_0, w_{A_2}) \\
&+ \sum_{i=1}^n D_{w_0, w_i} \omega_{g,n}(x, w_{[n]_i}).
\end{aligned}$$

Lemma 3.2. *For $m \geq 0$, one has*

$$D_{u,v} x^m = uv(u^m + 3u^{m-1}v + 5u^{m-2}v^2 + \dots + (2m+1)v^m).$$

Proof. This is elementary:

$$\begin{aligned}
& \frac{u(u+v)}{(u-v)^2} \cdot u^m + \frac{3v}{v-u} \cdot v^m - \frac{2v^2}{(v-u)^2} \cdot v^m + \frac{2v^2}{v-u} \cdot mv^{m-1} \\
&= \frac{u^{m+2} + u^{m+1}v - 2v^{m+2}}{(u-v)^2} - \frac{(2m+3)v^{m+1}}{u-v} \\
&= \frac{u^{m+1}(u-v) + 2(u^{m+1} - v^{m+1})v}{(u-v)^2} - \frac{(2m+3)v^{m+1}}{u-v} \\
&= \frac{1}{u-v} (u^{m+1} + 2(u^m + u^{m-1}v + \dots + v^{m-1})v - (2m+3)v^{m+1}) \\
&= \frac{1}{u-v} [(u^{m+1} - v^{m+1}) + 2(u^m - v^m)v + 2(u^{m-1} - v^{m-1})v^2 \\
&+ \dots + (u-v)v^m] \\
&= u^m + 3u^{m-1}v + 5u^{m-2}v^2 + \dots + (2m+1)v^m.
\end{aligned}$$

□

4. RECURSION RELATIONS FOR THE ANTIDERIVATIVES OF $\omega_{g,n}$

4.1. **The antiderivatives of $\omega_{g,n}$.** These are defined by:

$$(34) \quad \Omega_{g,n} = \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^n (2a_i - 1)!! w_i^{a_i+1/2}.$$

Here we use the following convention:

$$(35) \quad (-1)!! = 1.$$

For example,

$$\begin{aligned}
\Omega_{0,3}(w_1, w_2, w_3) &= (w_1 w_2 w_3)^{1/2}, \\
\Omega_{0,4}(w_1, \dots, w_4) &= w_1^{1/2} \cdots w_4^{1/2} \sum_{i=1}^4 w_i, \\
\Omega_{0,5}(w_1, \dots, w_4) &= (w_1 \cdots w_4)^{1/2} \left(\sum_{i=1}^4 3w_i^2 + \sum_{1 \leq i < j \leq 5} 2w_i w_j \right), \\
\Omega_{1,1}(w_1) &= \frac{1}{24} w_1^{3/2}, \\
\Omega_{1,2}(w_1, w_2) &= \frac{(w_1 w_2)^{1/2}}{24} \left(3w_1^2 + 3w_2^2 + w_1 w_2 \right), \\
\Omega_{1,3}(w_1, w_2, w_3) &= \frac{1}{24(\omega_1 \omega_2 \omega_3)^{1/2}} \cdot \left(15 \sum_{i=1}^3 \omega_i^3 + 6 \sum_{1 \leq i \neq j \leq 3} w_i^2 w_j + 2w_1 w_2 w_3 \right).
\end{aligned}$$

Lemma 4.1. *The functions $\omega_{g,n}$ and $\Omega_{g,n}$ are related by:*

$$(36) \quad \omega_{g,n} = 2^n \prod_{j=1}^n w_j^{3/2} \cdot \partial_{w_1} \cdots \partial_{w_n} \Omega_{g,n}.$$

Proof. This is easy to see from (28) and (34). \square

4.2. Recursion relations for $\Omega_{g,n}$. We need the following easy observation.

Lemma 4.2. *The following identity holds:*

$$(37) \quad \begin{aligned} & D_{w_0, w_i} \omega_{g,n}(x, w_{[n]_i}) \\ &= 2^{n+1} (w_1 \cdots w_n)^{3/2} \partial_{w_1} \cdots \partial_{w_n} \mathcal{D}_{w_0, w_i} \partial_x \Omega_{g,n}(x, w_{[n]_i}), \end{aligned}$$

where $\mathcal{D}_{u,v} : \mathbb{C}[x]x^{-1/2} \rightarrow \mathbb{C}[u,v]uv^{1/2}$ is a linear operator defined by:

$$(38) \quad \mathcal{D}_{u,v} x^{a-1/2} = uv^{1/2} (u^{a+1} + u^a v + \cdots + v^{a+1}).$$

Proof. By (28),

$$\begin{aligned} & \omega_{g,n}(x, w_{[n]_i}) \\ &= \sum_{a, a_1, \dots, \hat{a}_i, \dots, a_n \geq 0} \langle \tau_a \tau_{a_1} \cdots \widehat{\tau_{a_i}} \cdots \tau_{a_n} \rangle_g (2a+1)!! x^{a+1} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (2a_j+1)!! w_j^{a_j+1}, \end{aligned}$$

and so

$$\begin{aligned}
& D_{w_0, w_i} \omega_{g, n}(x, w_{[n]_i}) \\
&= \sum_{a, a_1, \dots, \hat{a}_i, \dots, a_n \geq 0} \langle \tau_a \tau_{a_1} \cdots \widehat{\tau_{a_i}} \cdots \tau_{a_n} \rangle_g (2a+1)!! \cdot D_{w_0, w_i} x^{a+1} \\
&\quad \cdot \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (2a_j + 1)!! w_j^{a_j+1} \\
&= \sum_{a, a_1, \dots, \hat{a}_i, \dots, a_n \geq 0} \langle \tau_a \tau_{a_1} \cdots \widehat{\tau_{a_i}} \cdots \tau_{a_n} \rangle_g \cdot \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (2a_j + 1)!! w_j^{a_j+1} \\
&\quad \cdot (2a+1)!! \cdot w_0 w_i (w_0^{a+1} + 3w_0^a w_i + \cdots + (2a+3)w_i^{a+1}) \\
&= 2^{n+1} (w_1 \cdots w_n)^{3/2} \partial_{w_1} \cdots \partial_{w_n} \sum_{a, a_1, \dots, \hat{a}_i, \dots, a_n \geq 0} \langle \tau_a \tau_{a_1} \cdots \widehat{\tau_{a_i}} \cdots \tau_{a_n} \rangle_g \\
&\quad \cdot \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (2a_j - 1)!! w_j^{a_j+1/2} \cdot (2a+1)!! \\
&\quad \cdot w_0 w_i^{1/2} (w_0^{a+1} + w_0^a w_i + \cdots + w_i^{a+1}) \\
&= 2^{n+1} (w_1 \cdots w_n)^{3/2} \partial_{w_1} \cdots \partial_{w_n} \mathcal{D}_{w_0, w_i} \partial_x \Omega_{g, n}(x, w_{[n]_i}).
\end{aligned}$$

□

With the above two Lemmas, it is easy to derive from Theorem

Theorem 4.3. *Except for the case of $(g, n) = (0, 2)$, the following recursion relations hold:*

$$\begin{aligned}
& \partial_{w_0} \Omega_{g, n+1}(w_0, w_1, \dots, w_n) \\
&= w_0^{5/2} \partial_x \partial_y \Omega_{g-1, n+2}(x, y, w_{[n]})|_{x=y=w_0} \\
(39) \quad & + w_0^{5/2} \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}}^s \partial_{w_0} \Omega_{g_1, |A_1|+1}(w_0, w_{A_1}) \cdot \partial_{w_0} \Omega_{g_2, |A_2|+1}(w_0, w_{A_2}) \\
& + w_0^{-3/2} \sum_{i=1}^n \mathcal{D}_{w_0, w_i} \partial_x \Omega_{g, n}(x, w_{[n]_i}).
\end{aligned}$$

4.3. Examples.

4.3.1. *The $(g, n) = (0, 3)$ case.*

$$\begin{aligned}
\partial_{w_0} \Omega_{0,4}(w_0, w_1, w_2, w_3) &= w_0^{-3/2} \sum_{i=1}^3 \mathcal{D}_{w_0, w_i} \partial_x \Omega_{0,3}(x, w_{[3]_i}) \\
&= w_0^{-3/2} \sum_{i=1}^3 \mathcal{D}_{w_0, w_i} \partial_x (x^{1/2} \cdot \frac{(w_1 w_2 w_3)^{1/2}}{w_i^{1/2}}) \\
&= \frac{1}{2} w_0^{-3/2} \sum_{i=1}^3 w_0 w_i^{1/2} (w_0 + w_i) \cdot \frac{(w_1 w_2 w_3)^{1/2}}{w_i^{1/2}} \\
&= \frac{3}{2} w_0^{1/2} (w_1 w_2 w_3)^{1/2} + \frac{1}{2} w_0^{-1/2} (w_1 w_2 w_3)^{1/2} \sum_{i=1}^3 w_i.
\end{aligned}$$

4.3.2. *The $(g, n) = (1, 1)$ case.*

$$\begin{aligned}
\partial_{w_0} \Omega_{1,2}(w_0, w_1) &= w_0^{5/2} \partial_x \partial_y \Omega_{0,3}(x, y, w_1)|_{x=y=w_0} + u^{-3/2} \mathcal{D}_{w_0, w_1} \Omega_{1,1}(x) \\
&= w_0^{5/2} \cdot \partial_x \partial_y ((xy w_1)^{1/2})|_{x=y=w_0} + \mathcal{D}_{w_0, w_1} \partial_x \frac{1}{24} x^{3/2} \\
&= \frac{1}{4} w_0^{3/2} w_1^{1/2} + \frac{1}{16} w_0^{-3/2} \cdot w_0 w_1^{1/2} (w_0^2 + w_0 w_1 + w_1^2) \\
&= \frac{1}{16} w_0^{-1/2} w_1^{1/2} (5w_0^2 + w_0 w_1 + w_1^2).
\end{aligned}$$

5. GUKOV-SULKOWSKI CONJECTURE FOR THE AIRY CURVE

5.1. **The formulation of the conjecture in u -coordinates.** Quantization of the defining polynomial of the Airy curve

$$(40) \quad A = \frac{1}{2} v^2 - u$$

by the assignment

$$(41) \quad \hat{u} = u, \quad \hat{v} = \hbar \partial_u.$$

yields the following differential operator:

$$(42) \quad \hat{A} = \frac{1}{2} \hbar^2 \partial_u^2 - u.$$

Under the following parametrization

$$(43) \quad u(z) = \frac{1}{2} z^2,$$

$$(44) \quad v(z) = z,$$

we have

$$(45) \quad S_0 = \int^z v(z) du(z) = \int^z z^2 dz = \frac{1}{3} z^3,$$

$$(46) \quad S_1 = -\frac{1}{2} \log \frac{du}{dz} = -\frac{1}{2} \log(z),$$

and

$$(47) \quad \begin{aligned} S_n &= \sum_{2g-1+k=n} \frac{(-1)^k}{k!} \int^z dz'_1 \cdots \int^z dz'_n \mathcal{W}_{g,k}(z'_1, \dots, z'_k) \\ &= \sum_{2g-1+k=n} \frac{(-1)^k}{k!} \Xi_{g,k}(z, \dots, z), \end{aligned}$$

where

$$(48) \quad \Xi_{g,n}(z_1, \dots, z_n) = \int^{z_1} \cdots \int^{z_n} \mathcal{W}_{g,n}(z_1, \dots, z_n) dz_1 \cdots dz_n$$

$$(49) \quad = (-1)^n \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^n \frac{(2a_i - 1)!!}{z_i^{2a_i+1}}.$$

For example,

$$S_2 = -\Xi_{1,1}(z) - \frac{1}{3!} \Xi_{0,3}(z, z, z) = \frac{1}{24z^3} + \frac{1}{6z^3} = \frac{5}{24z^3}.$$

$$S_3 = \frac{1}{2!} \Xi_{1,2}(z, z) + \frac{1}{4!} \Xi_{0,4}(z, z, z, z) = \frac{1}{2} \cdot \frac{7}{24z^6} + \frac{1}{24} \cdot \frac{4}{z^6} = \frac{5}{16z^6}.$$

Choose $z = u^{1/2}$ or $z = -u^{1/2}$, one then expresses S_n in the u -coordinates. For example,

$$(50) \quad S_0 = \pm \frac{1}{3} (2u)^{3/2},$$

$$(51) \quad S_1 = -\frac{1}{4} \log(2u) + \text{constant},$$

$$(52) \quad S_2 = \pm \frac{5}{24(2u)^{3/2}},$$

$$(53) \quad S_3 = \frac{5}{16} \frac{1}{(2u)^3}.$$

Recall

$$(54) \quad Z = \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n.$$

Therefore,

$$\begin{aligned}
\hat{A}Z &= \left(\frac{1}{2}\hbar^2\partial_u^2 - u\right) \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n \\
&= \frac{1}{2}\hbar\partial_u \left(\sum_{n=0}^{\infty} \hbar^n \partial_u S_n \cdot \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n \right) - u \cdot \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n \\
&= \left(\frac{1}{2} \left(\sum_{n=0}^{\infty} \hbar^n \partial_u S_n \right)^2 + \frac{1}{2} \sum_{n=0}^{\infty} \hbar^{n+1} \partial_u^2 S_n - u \right) \cdot \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n \\
&= \left(\left(\frac{1}{2} (\partial_u S_0)^2 - u \right) + \sum_{n=1}^{\infty} \hbar^n \left(\frac{1}{2} \partial_u^2 S_{n-1} + \frac{1}{2} \sum_{i+j=n} \partial_u S_i \cdot \partial_u S_j \right) \right) \cdot Z.
\end{aligned}$$

It follows that the equation

$$(55) \quad \hat{A}Z = 0$$

is equivalent to the following sequence of equations:

$$(56) \quad \frac{1}{2}(\partial_u S_0)^2 = u,$$

$$(57) \quad \frac{1}{2}\partial_u^2 S_0 + \partial_u S_0 \cdot \partial_u S_1 = 0,$$

$$(58) \quad \frac{1}{2}\partial_u^2 S_1 + \partial_u S_0 \cdot \partial_u S_2 + \frac{1}{2}\partial_u S_1 \cdot \partial_u S_1 = 0,$$

$$(59) \quad \frac{1}{2}\partial_u^2 S_{n-1} + \partial_u S_0 \cdot \partial_u S_n + \partial_u S_1 \cdot \partial_u S_{n-1} + \frac{1}{2} \sum_{\substack{i+j=n \\ i,j \geq 2}} \partial_u S_i \cdot \partial_u S_j = 0,$$

where $n > 2$. One can plug in (50)-(52) to check that (56)-(58) hold and one can rewrite (59) as follows:

$$(60) \quad \partial_u S_n = \pm \frac{1}{2(2u)^{1/2}} \left(-\partial_u^2 S_{n-1} + \frac{1}{2u} \partial_u S_{n-1} - \sum_{\substack{i+j=n \\ i,j \geq 2}} \partial_u S_i \cdot \partial_u S_j \right).$$

It suffices to prove this for $n \geq 3$.

5.2. The reformulation in w -coordinates. Because

$$w(z) = \frac{1}{z^2},$$

we have

$$\begin{aligned}
w &= \frac{1}{2u}, \\
\partial_u &= -2w^2 \partial_w, \\
\partial_u^2 &= 4w^4 \partial_w^2 + 8w^3 \partial_w.
\end{aligned}$$

Hence one can rewrite (61)

$$(61) \quad \partial_u S_n = \pm \frac{1}{2(2u)^{1/2}} \left(-\partial_u^2 S_{n-1} + \frac{1}{2u} \partial_u S_{n-1} - \sum_{\substack{i+j=n \\ i,j \geq 2}} \partial_u S_i \cdot \partial_u S_j \right).$$

as follows:

$$\begin{aligned} -2w^2 \partial_w S_n &= \pm \frac{w^{1/2}}{2} \left(-(4w^4 \partial_w^2 + 8w^3 \partial_w) S_{n-1} \right. \\ &\quad \left. - \frac{w}{2} \cdot 2w^2 \partial_w S_{n-1} - \sum_{\substack{i+j=n \\ i,j \geq 2}} 2w^2 \partial_w S_i \cdot 2w^2 \partial_w S_j \right). \end{aligned}$$

After simplification one gets

$$(62) \quad \partial_w S_n = \pm \left(\frac{5}{2} w^{3/2} \partial_w S_{n-1} + w^{5/2} \cdot \partial_w^2 S_{n-1} + w^{5/2} \sum_{\substack{i+j=n \\ i,j \geq 2}} \partial_w S_i \cdot \partial_w S_j \right),$$

or equivalently,

$$(63) \quad w^{5/2} \partial_w S_n = \pm \left((w^{5/2} \partial_w)^2 S_{n-1} + \sum_{\substack{i+j=n \\ i,j \geq 2}} w^{5/2} \partial_w S_i \cdot w^{5/2} \partial_w S_j \right).$$

Remark 5.1. Let $t = -\frac{2}{3}w^{-3/2}$, then one has

$$(64) \quad \partial_w = \frac{\partial t}{\partial w} \partial_t = w^{-5/2} \partial_t,$$

and so

$$(65) \quad \partial_t S_n = \partial_t^2 S_{n-1} + \sum_{\substack{i+j=n \\ i,j \geq 2}} \partial_t S_i \cdot \partial_t S_j.$$

5.3. The proof of Theorem. It suffices now to establish (63). By comparing with (34), we have

$$(66) \quad \Omega_{g,n}(w_1, \dots, w_n) = (\mp 1)^n \Xi_{g,n}(z_1, \dots, z_n),$$

and so

$$\begin{aligned} (67) \quad S_n &= \sum_{2g-1+k=n} \frac{(-1)^k (\mp 1)^k}{k!} \Omega_{g,k}(w, \dots, w) \\ &= (\pm 1)^{n+1} \sum_{2g-1+k=n} \frac{1}{k!} \Omega_{g,k}(w, \dots, w). \end{aligned}$$

By (67),

$$w^{5/2}\partial_w S_n = (\pm 1)^{n+1} \sum_{2g-1+k=n} \frac{1}{(k-1)!} w^{5/2}\partial_w \Omega_{g,k}(w, \dots, w),$$

and

$$\begin{aligned} (w^{5/2}\partial_w)^2 S_n &= (\pm 1)^{n+1} \sum_{2g-1+k=n} \frac{1}{(k-1)!} (x^{5/2}\partial_x)^2 \Omega_{g,k}(x, w, \dots, w)|_{x=w} \\ + (\pm 1)^{n+1} \sum_{2g-1+k=n} \frac{1}{(k-2)!} x^{5/2}\partial_x y^{5/2}\partial_y \Omega_{g,k}(x, y, w, \dots, w)|_{x=y=w}. \end{aligned}$$

Hence (63) can be rewritten as follows (after taking care of the \pm signs):

$$\begin{aligned} & \sum_{2g-1+k=n} \frac{1}{(k-1)!} w^{5/2}\partial_w \Omega_{g,k}(w, \dots, w) \\ &= \sum_{2g-1+k=n-1} \frac{1}{(k-1)!} (x^{5/2}\partial_x)^2 \Omega_{g,k}(x, w, \dots, w)|_{x=w} \\ (68) \quad & + \sum_{2g-1+k=n-1} \frac{1}{(k-2)!} x^{5/2}\partial_x y^{5/2}\partial_y \Omega_{g,k}(x, y, w, \dots, w)|_{x=y=w} \\ & + w^{5/2} \sum_{\substack{i+j=n \\ i,j \geq 2}} \sum_{2g_1-1+k_1=i} \frac{1}{(k_1-1)!} w^{5/2}\partial_w \Omega_{g_1,k_1}(w, \dots, w) \\ & \cdot \sum_{2g_2-1+k_2=j} \frac{1}{(k_2-1)!} w^{5/2}\partial_w \Omega_{g_2,k_2}(w, \dots, w). \end{aligned}$$

We now show that this can be derived from (39). We first set $n = k-1$ on both sides of (39) to get:

$$\begin{aligned} (69) \quad & \omega_0^{5/2}\partial_{w_0} \Omega_{g,k}(w_0, w_1, \dots, w_{k-1}) \\ &= x^{5/2}\partial_x y^{5/2}\partial_y \Omega_{g-1,k+1}(x, y, w_{[k-1]})|_{x=y=w_0} \\ & + \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[k-1]}}^s w_0^{5/2}\partial_{w_0} \Omega_{g_1,|A_1|+1}(w_0, w_{A_1}) \cdot w_0^{5/2}\partial_{w_0} \Omega_{g_2,|A_2|+1}(w_0, w_{A_2}) \\ & + \sum_{i=1}^{k-1} w_0 \mathcal{D}_{w_0, w_i} \partial_x \Omega_{g,k-1}(x, w_{[k-1]_i}). \end{aligned}$$

We then set $w_0 = \dots = w_{k-1} = w$ on both sides of (69). The left-hand side becomes:

$$w^{5/2} \partial_w \Omega_{g,k}(w, \dots, w).$$

The first line on the right-hand side becomes:

$$x^{5/2} \partial_x y^{5/2} \partial_y \Omega_{g-1,k+1}(x, y, w, \dots, w)|_{x=y=w}.$$

The second line on the right-hand side becomes:

$$w^{5/2} \sum_{\substack{g_1+g_2=g \\ k_1+k_2=k+1}}^s \frac{(k-1)!}{k_1!k_2!} \partial_w \Omega_{g_1,k_1}(w, \dots, w) \cdot \partial_w \Omega_{g_2,k_2}(w_0, \dots, w).$$

Now we deal with the third line on the right-hand side. Recall

$$(70) \quad \mathcal{D}_{u,v} x^{a-1/2} = uv^{1/2}(u^{a+1} + u^a v + \dots + v^{a+1}).$$

So we have

$$\begin{aligned} w_0 \mathcal{D}_{w_0, w_i} \partial_x x^{a+1/2} &= w_0 \mathcal{D}_{w_0, w_i} (a+1/2) x^{a-1/2} \\ &= (a+1/2) w_0 \cdot w_0 w_i^{1/2} (w_0^{a+1} + w_0^a w_i + \dots + w_i^{a+1}), \end{aligned}$$

it follows that

$$\begin{aligned} w_0 \mathcal{D}_{w_0, w_i} \partial_x x^{a+1/2}|_{w_0=w_i=w} &= (a+2)(a+1/2) w \cdot w w^{1/2} w^{a+1} \\ &= (a+2)(a+1/2) w^{a+7/2}. \end{aligned}$$

On the other hand

$$\begin{aligned} (w^{5/2} \partial_w)^2 w^{a+1/2} &= w^{5/2} \partial_w (w^{5/2} \cdot (a+1/2) w^{a-1/2}) \\ &= (a+1/2) w^{5/2} \partial_w w^{a+2} \\ &= (a+1/2)(a+2) w^{5/2} w^{a+1} \\ &= (a+1/2)(a+2) w^{a+7/2}. \end{aligned}$$

Hence we get:

$$\begin{aligned} &\sum_{i=1}^{k-1} w_0 \mathcal{D}_{w_0, w_i} \partial_x \Omega_{g,k-1}(x, w_{[k-1]_i})|_{w_0=\dots=w_{k-1}=w} \\ &= (k-1)(x^{5/2} \partial_x)^2 \Omega_{g,k-1}(x, w, \dots, w)|_{x=w}. \end{aligned}$$

To summarize, we have obtained the following identity:

$$\begin{aligned}
& w^{5/2} \partial_w \Omega_{g,k}(w, \dots, w) \\
&= x^{5/2} \partial_x y^{5/2} \partial_y \Omega_{g-1,k+1}(x, y, w, \dots, w)|_{x=y=w} \\
&+ w^{5/2} \sum_{\substack{g_1+g_2=g \\ k_1+k_2=k+1}}^s \frac{(k-1)!}{k_1!k_2!} \partial_w \Omega_{g_1,k_1}(w, \dots, w) \cdot \partial_w \Omega_{g_2,k_2}(w, \dots, w) \\
&+ (k-1)(x^{5/2} \partial_x)^2 \Omega_{g,k-1}(x, w, \dots, w)|_{x=w}.
\end{aligned}$$

Dividing both sides by $\frac{1}{(k-1)!}$ and take $\sum_{2g-1+k=n}$, one gets (68). This completes the proof.

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